Editors' Introduction: What Is Mathematical Visualization?[1]

Walter Zimmermann[2] and Steve Cunningham[3]

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In the preface to *Geometry and the Imagination* [1], Hilbert wrote:

"in mathematics ... we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations.

"...With the aid of visual imagination [*Anschauung*] we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof ... In this manner, geometry being as many faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains."

The Visualization Renaissance

This passage effectively captures the spirit and purpose of this volume. Paraphrasing Hilbert, it is our goal to explore how "with the aid of visual imagination" one can "illuminate the manifold facts and problems of mathematics." We take the term *visualization* to describe the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated. Our goal is to provide an overview of research, analysis, practical experience, and informed opinion about visualization and its role in teaching and learning mathematics, especially at the undergraduate level.

The sciences, engineering, and to a more limited extent mathematics are enjoying a renaissance of interest in visualization. To a significant degree, this renaissance is being driven by technological developments. Computer graphics has greatly expanded the scope and power of visualization in every field. A recent report of the Board on Mathematical Sciences of the National Research Council [2] identifies a number of recent research accomplishments and related opportunities in mathematics. One of the areas cited is "Computer Visualization as a Mathematical Tool." According to this report, "in recent years

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[1] *Visualization in Teaching and Learning Mathematics.* Walter Zimmermann & Steve Cunningham Eds. MAA Notes Number 19, 1991.

[2] *Walter Zimmemann is Professor and Chair of the Department of Mathematics at the University of the Pacific. He is an author of educational software for mathematics, and is a member of the Committee on Computers in Mathematics Education of the MAA.*

[3] *Steve Cunningham is Professor of Computer Science at California State University Stanislaus. He has written widely on issues in computer graphics for education and is actively involved in both computer science and computer graphics education.*

computer graphics have played an increasingly important role in both core and applied mathematics, and the opportunities for utilization are enormous."

A report to the National Science Foundation, *Visualization in Scientific Computing* (VISC) [3], calls for a national initiative "to get visualization tools into the 'hands and minds' of scientists." The report asserts that "Visualization ... transforms the symbolic into the geometric, enabling researchers to observe their simulations and computations. Visualization offers a method of seeing the unseen. It enriches the process of scientific discovery and fosters profound and unexpected insights. In many fields it is already revolutionizing the way scientists do science." The focus of the VISC report is on the applied sciences, not mathematics. However, we would argue that much of the rationale of the report applies to mathematics as well as the sciences, and to teaching as well as research.

Visualization and the Nature of Mathematics

Hilbert refers to the two "tendencies" present in mathematics, logic and intuition. In their paper, "Nonanalytic Aspects of Mathematics and Their Implications for Research and Education," Philip J. Davis and James A. Anderson [4] observe: "Mathematics has elements that are spatial, kinesthetic, elements that are arithmetic or algebraic, elements that are verbal, programmatic. It has elements that are logical, didactic and elements that are intuitive, or even counter-intuitive... These may be compared to different modes of consciousness. To place undue emphasis on one element or group of elements upsets a balance. It results in an impoverishment of the science and represents an unfulfilled potential. ...We must not block off any mode of experience or thought." They go on to make a series of suggestions for implementing this philosophy. These recommendations include (among others): "Restore geometry. Restore intuitive and experimental mathematics. Give a proper place to computing and programmatics. Make full use of computer graphics." This volume is conceived very much in the spirit of these recommendations.

Visualization is not a recent invention. In "Picture Puzzling," Ivan Rival [5] writes, "Diagrams are, of course, as old as mathematics itself. Geometry has always relied heavily on pictures, and, for a time, other branches of mathematics did too. Even Isaac Newton ... did not actually prove [the] fundamental theorems. ...Had you asked him to justify them, he would likely have presented an argument that, though compelling, was loose and depended heavily on pictures."

Rival goes on to observe that "the intuitive and often persuasive style of argument used by Newton and his contemporaries fell into disrepute during the nineteenth century after it proved, in several celebrated cases, to be misleading." Davis and Anderson (in the paper quoted above) describe this process by the colorful phrase, "The Degradation of the Geometric Consciousness." They observe that "...over the past century and a half there has been a steady and progressive degradation of the geometric and kinesthetic elements of mathematical instruction and research. During this period the formal, the symbolic, the verbal, the analytic elements have prospered greatly."

The Science of Patterns

There is evidence that the pendulum has recently begun to swing back to a more balanced view of mathematics which takes into account more fully the visual and intuitive dimensions. In the article cited above, Rival states that "the limits of deductivism are at last dawning on mathematicians, thanks largely to computers." The role of computers in reshaping our notions of mathematics is echoed by Lynn Steen. In "The Science of Patterns" [6], Steen writes, "Mathematics is often defined as the science of space and number, as the discipline rooted in geometry and arithmetic. Although the diversity of modern mathematics has always exceeded this definition, it was not until the recent resonance of computers and mathematics that a more apt definition became fully apparent. Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to 'explain' and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns."

In speaking of "patterns," Steen is speaking metaphorically, but the metaphor of patterns is surely a visual metaphor. Not all patterns can be visualized, but it is as natural to want to visualize a pattern as it is to want to hear a melody. If mathematics is the science of patterns, it is natural to try to find the most effective ways to visualize these patterns and to learn to use visualization creatively as a tool for understanding. This is the essence of mathematical visualization.

Visualization and Underftanding

The term "visualization" is unfamiliar to some in the context of mathematics, and its connotations may not be obvious. Our use of the term differs somewhat from common usage in everyday speech and in psychology, where the meaning of visualization is closer to Its fundamental meaning, "to form a mental image." For example, there are psychological studies which focus on the subject's ability to form and manipulate mental images. In these studies, there is no question of using pencil and paper, much less a computer, to answer the questions. From the perspective of mathematical visualization, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualization what we are interested in is precisely the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagram to achieve understanding, and as an aid in problem solving. In mathematics, visualization is not an end in itself but a means toward an end, which is understanding. Notice that, typically, one does not speak about visualizing a *diagram* but visualizing a *concept or problem.* To visualize a diagram means simply to form a mental image of the diagram, but to visualize a problem means to understand the problem in terms of a diagram or visual image. Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding.

Anschauung *and intuition.* The title of Hilbert's book, quoted at the beginning of this article, is *Anschauliche Geometrie* [7]. The complexity (and ambiguity) of the term "Visualization" reflects the richness of the related German noun *Anschauung,* from which the adjective *anschaulich* is derived. In *Imagery in Scientific Thought* [8], Miller points out that etymologically, *Anschauung* refers to visual perception, but Kant extended its use to cover any sort of perception, and subsequently the term took on a meaning often translated as *intuition.* In one context, Miller says, '...*Anschauung* refers to the intuition through pictures formed in the mind's eye." and "*Anschauung* is superior to viewing merely with the senses." (p. 110). In another context, *Anschauung* is "knowledge obtained by contemplation of ideas already in the mind..." (p. 274). Miller does not address mathematical visualization directly, but we would like to borrow his language to say that mathematical visualization also involves "intuition through pictures formed in the mind's eye." Thus *Anschauliche Geometrie,* literally rendered *visualizable geometry,* is also *intuitive geometry,* and by implication visualizable mathematics is also intuitive mathematics.

The VISC report points out that "An estimated 50 percent of the brain's neurons are associated with vision. Visualization in scientific computing aims to put that neurological machinery to work." This interesting statement nevertheless blurs an important distinction because of the way it slips from *vision to visualization.* Miller's observation that "*Anschauung* is superior to viewing merely with the senses," implies that visualization is superior to vision. Visualization implies understanding - the kind of understanding that comes from "intuition through pictures formed in the mind's eye." Furthermore, in mathematics as well as in scientific computing, one may visualize something that is not seen or may never have been seen. Visualization may be "knowledge obtained by contemplation of ideas already in the mind... ." Many mathematicians can recall the experience of having an image come spontaneously to mind in the course of solving a problem - an image of some object or figure which they may have never actually seen.

Mathematical visualization is not "math appreciation through pictures." The intuition which mathematical visualization seeks is not a vague kind of intuition, a superficial substitute for understanding, but the kind of intuition which penetrates to the heart of an idea. It gives depth and meaning to understanding, serves as a reliable guide to problem solving, and inspires creative discoveries. To achieve this kind of understanding, visualization cannot be isolated from the rest of mathematics. Visual thinking and graphical representations must be linked to other modes of mathematical thinking and other forms of representation. One must learn how ideas can be represented symbolically, numerically, and graphically, and to move back and forth among these modes. One must develop the ability to choose the approach most appropriate for a particular problem, and to understand the limitations of these three dialects of the language of mathematics. We have thus strongly encouraged the authors of these papers to show how visualization operates in a mathematical context and not as an isolated topic.

Computing and Visualization

Steen and others have noted the role of computing in reshaping our concept of the nature of mathematics. Computers have a direct and concrete role in this visualization renaissance because of the ways computers can generate mathematical graphics. The beautiful and complex fractal images with which we are now familiar come to mind, but the images that come from mathematical visualization are varied indeed. They include geometrical figures of all kinds in two or three dimensions; they include curves and surfaces, direction fields, contour plots and other similar figures; they include graphs (in the sense of graph theory) and other kinds of schematic diagrams, such as Venn diagrams. The images being described need not be static; they may be dynamic or interactive (user controlled). Graphic simulations of processes, real or hypothetical, may belong to mathematical visualization, regardless of the nature of the images.

To encompass the whole scope of visualization, one should consider other technologies such as videotape, film, and interactive videodisc. These technologies further expand the kinds of images that can be used. Visualization technology is evolving so rapidly that it is impossible at this time to sketch its boundaries. The same is true about mathematical applications of visualization. Who could have predicted the explosion of interest in chaotic dynamics and the associated imagery of fractals? What will tomorrow bring?

There are many important questions about the role of computers in visualization. How can the power of computers in general, and interactive computer graphics in particular, be used most effectively to promote mathematical insight and understanding? What are the characteristics of good educational software? What are the roles of classroom demonstrations, structured computer laboratory exercises, and free exploration of mathematical ideas? What will be the impact of computers on the mathematics curriculum? In other words, how can computers help us to teach what we now teach more effectively, and what new problems, topics, or fields of mathematics are, opened up by the new technology? Many of these questions are addressed, directly or indirectly, in the papers presented here.

Computer-based visualization, whether static, dynamic, or interactive, is only one facet of the role of computers in mathematics. Visualization must be linked to the numerical and symbolic aspects of mathematics to achieve the greatest results. Some of the most interesting and important applications of visualization involve problems which use numerical and/or symbolic processing as well as graphics.

A Paradigm for Using Color Graphics. Recent developments in the theory of nonlinear dynamical systems have not only dramatized the power of computer graphics but have established a new paradigm for mathematical visualization. Because of the crucial role of fractal images and other graphics in the study of dynamical systems, this subject has become a model, in general and specific terms, for other fields. For example, this model is directly acknowledged in a recent paper by Coffey, et al. [9], who write: "Several developments underlie the present revival in classical mechanics, including computerized algebraic processors and color graphics. ... Color graphics proves invaluable in visualizing the global behavior and discovering minutiae of local behavior hidden beneath the mass of calculation. Pseudocoloring a function over a domain, a widespread technique in applied mathematics, has produced stunning pictures; they have opened the eyes of mathematicians to hitherto unsuspected phenomena in the dynamics of nonlinear maps. Extension to classical mechanics forces a search for refinements in the technique such as automatic selection of colors to ensure enough contrast around isolated but dose singularities."

Because of the importance of computers in visualization, it is natural to identify the field of visualization exclusively with computer-based visualization. In the context of mathematics, this would be a mistake. We construe the field of mathematical visualization broadly to include non computer-based visualization as well as visualization based on computers or other technologies. The ability to draw a simple figure to represent a mathematical problem, to interpret such figures with understanding, and to use such figures as an aid in problem solving are fundamental visualization skills. Without such fundamental skills, it is unlikely that computer-based visualization can be used efficiently, or even meaningfully. Vision is not visualization; to see is not necessarily to understand.

The Origins and Objectives of this Volume

The literature of mathematics includes works which address the graphical representation of mathematical ideas. One noteworthy recent example is George Francis' *A Topological*

Picturebook [10]. Significantly, Francis' inspiration did not come from modern computer graphics but from the German *Enzyclopaedie der Mathematischen Wissenschaften* and the collected works of Felix Klein. He writes, “Theirs was a wonderfully straightforward way of looking at rather complicated things, notably Riemann surfaces and geometrical constructions over the complex numbers. They drew pictures, built models and wrote manuals on how to do this. And so they also captured a vivid record of the mathematics of their day.”

The tradition of the *Enzyclopaedie* has been nearly lost, and works such as Francis' are rare. When visualization is discussed today, it is usually in the context of another subject. Sometimes, graphics is treated as one of the features of so-called “computer algebra systems” (CAS) and discussions of visualization are subsumed under the rubric of “symbolic computation.” We consider this practice unfortunate. Symbolic computation, numerical computation and graphics are three complementary modes of mathematical computing, and to refer to computer programs which have all these capabilities as "computer algebra systems' is misleading. Terminology aside, we believe that visualization involves an interesting and distinctive set of mathematical, pedagogical and practical questions which ought to be addressed directly, not just as a side-issue in another context. This belief led us to undertake the publication of this volume.

The Organization of this Volume. This volume explores the role of visualization in mathematics education, especially undergraduate education. The first five papers address fundamental visualization issues which cut across different subject fields. The remaining papers address the role of visualization in particular fields of mathematics - geometry, calculus, differential equations, differential geometry, linear algebra, numerical analysis, complex analysis, stochastic processes and statistics. Some of these papers also address general visualization issues as they arise in the context of their fields. With two exceptions (the papers by Goldenberg), the papers are original papers prepared for this volume. These two papers are included at the request of the editors (and, of course, with permission), because of the contribution we feel they make.

There are few clear precedents or contemporary standards for publications in the field of mathematical visualization. We exercised our best judgment in selecting and refining the papers which appear here, and we accept responsibility for the tone and scope of the volume. While these papers have mathematical content, they are not mathematics papers in the traditional sense, nor are they research papers in education. The tone of the papers, on the whole, is informal, and we consider this to be appropriate for the subject matter and to be consistent with the tradition of the *MAA Notes.* Unlike formal scholarly papers, some of these papers include opinion and speculation. This, too, we consider appropriate, as the field is rapidly evolving.

An issue we had to face was how to deal with papers which discuss software, especially software developed by the authors. It is natural for someone who has interests in visualization to be drawn to software development, and those who have developed and implemented software are among those best

qualified to discuss visualization. We considered it inappropriate to include papers whose goal seemed to be promotion of particular software. However, we felt it would be counterproductive to exclude papers just because they discuss the author's software. We made a conscious decision to include such papers where the focus of the paper is on mathematical or pedagogical issues, not on the software, and where any discussion of the software is in the context of such issues. Several papers in this volume include such discussion, and we are satisfied that they meet these criteria.

Broadly speaking, our goals in this project were to present ideas and information about the current state of mathematical visualization, to stimulate thought and discussion, and to provide a frame of reference for future efforts. We hope that this volume will contribute to a better understanding of the educational aspects of the field we have called mathematical visualization, and a broader recognition of its importance. We believe the papers should be judged by the extent to which they show how one can use visual imagination and visual thinking to "illuminate the manifold facts and problems" of mathematics. Do these papers, and the visual images they present, convey the "patterns" of mathematics? To what extent do these papers show how one can promote the development of intuition "Through pictures formed in the mind's eye"?

Visualization, like geometry, is "related to the most diverse branches of mathematics." Visualization is multifaceted: rooted in mathematics, the field has important historical, philosophical, psychological, pedagogical and technological aspects. If present trends are any indication, it seems that mathematics will evolve in a direction which will make visualization even more important in the future than it is now. At the same time, the evolution of technology will make more and more powerful visualization tools available. We hope all readers will find ideas here which they can use in their own work and in their classes, and we hope some readers will be encouraged to pursue studies in this fascinating subject. The opportunities for creative efforts by mathematicians and others appear to be unlimited.

References

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[10] Francis, George K., *A Topologkal Picturebook,* Springer, New York, 1987.